Thermodynamics of Taub-NUT/Bolt-AdS Black Holes in Einstein-Gauss-Bonnet Gravity

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Abstract

We give a review of the existence of Taub-NUT/bolt solutions in Einstein Gauss-Bonnet gravity with the parameter α in six dimensions. Although the spacetime with base space $S^2 \times S^2$ has curvature singularity at $r = N$, which does not admit NUT solutions, we may proceed with the same computations as in the $\mathbb{CP}^2$ case. The investigation of thermodynamics of NUT/Bolt solutions in six dimensions is carried out. We compute the finite action, mass, entropy, and temperature of the black hole. Then the validity of the first law of thermodynamics is demonstrated. It is shown that in NUT solutions all thermodynamic quantities for both base spaces are related to each other by substituting $\alpha^{\mathbb{CP}^k} = [(k+1)/k]\alpha^{S^2 \times S^2 \times \cdots S_k^2}$. So no further information is given by investigating NUT solution in the $S^2 \times S^2$ case. This relation is not true for bolt solutions. A generalization of the thermodynamics of black holes to arbitrary even dimensions is made using a new method based on the Gibbs-Duhem relation and Gibbs free energy for NUT solutions. According to this method, the finite action in Einstein Gauss-Bonnet is obtained by considering the generalized finite action in Einstein gravity with an additional term as a function of α. Stability analysis is done by investigating the heat capacity and entropy in the allowed range of α, Λ and N. For NUT solutions in d dimensions, there exist a stable phase at a narrow range of α. In six-dimensional Bolt solutions, metric is completely stable for $\mathcal{B} = S^2 \times S^2$, and is completely unstable for $\mathcal{B} = \mathbb{CP}^2$ case.

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I. INTRODUCTION

According to the general relativity principle, the most general classical theory of gravitation in d dimensions is Lovelock gravity. It is a higher dimensional generalization of Einstein gravity which can extend gravity to other consistent theories such as string theory. An interesting case which is second order Lovelock gravity is Gauss-Bonnet (GB) theory with $d \geq 5$ dimensions. In string theory context, when assuming that the tension of a string is large as compared to the energy scale of other variables, GB term is the first curvature correction to gravitation [1, 2]. Exactly the same as Einstein gravity, the Einstein-GB (EGB) field equation contains up to second order of metric tensor and is free of ghost. This theory has been extended to AdS and dS spacetimes [3]. Taub-NUT/Bolt-AdS (TN/BAdS) is a class of interesting spacetimes with negative cosmological constant which were investigated in EGB gravity [4].The original four-dimensional solution of Taub-NUT/Bolt spacetime [5, 6] is only locally asymptotically flat. The spacetime develops a boundary at infinity by twisting S^1 bundle over S^2, instead of simply being $S^1 \times S^2$. The boundary properties of TNAdS spacetimes has been discussed in Einstein gravity [7]. In general, the Killing vector which corresponds to the coordinate that parameterizes the fibre S^1 can have a zero-dimensional fixed point set (called a NUT solution) or a two-dimensional fixed point set (referred to as a 'Bolt' solution). Generalizations to higher dimensions and higher derivative gravity have been done [4, 8–14]. These solutions of Einstein gravity play a central role in the construction of M-theory configurations. Indeed, the 4-dimensional TNAdS solution provided the first test for the AdS/CFT correspondence in spacetimes that are only locally asymptotically AdS [15, 16], and the Taub-NUT metric is central to the supergravity realization of the $D6$-brane of type IIA string theory [17]. It is therefore natural to suppose that the generalization of these solutions to the case of EGB gravity, which is the low energy limit of supergravity, might provide us with a window into some interesting new corners of M-theory moduli space.

The AdS/CFT correspondence, is a famous conjecture that relates a $(d-1)$-dimensional conformal field theory to d-dimensional gravity theory with a negative cosmological constant [18]. In light of this conjecture, similar to quantum field theory, there are some counterterms which remove infinities in the gravity action. The counterterm contains some terms which are functions of the curvature invariants of the induced metric on the boundary. For

EGB gravity, similar to Einstein gravity [19–21], in computing the action and total mass, one encounters infrared divergences associated with the infinite volume of the spacetime. Recently, the counterterm for EGB gravity was obtained for $d \leq 9$ dimensions and applied to static and rotating black objects [22]. Also a counterterm for charged black holes (bhs) in GB gravity was proposed [23]. However, an alternative regularization prescription with any odd-even dimensions of AdS asymptotic spacetimes for any Lovelock theory has been proposed [24]. This approach is known as Kounterterms and uses boundary terms with explicit dependence on the extrinsic curvature K_{ab}.

The thermodynamics of black holes in Lovelock gravity attracted a lot of attention in the last decade [25]. In dealing with this subject, a regularized action can be obtained by using the counterterm renormalization method. For Taub-NUT/Bolt black holes the area law of entropy is broken and one may obtain the entropy by using Gibbs-Duhem relation [26]. Thermodynamics of Taub-NUT/Bolt black holes is investigated in Einstein and Einstein-Maxwell gravity in [9, 12, 27]. By extending the dimension of spacetimes higher than seven, one encounters so many terms in the counterterm that the computation of thermodynamic quantities would be very difficult. Also a counterterm for $d > 9$ in EGB gravity has not been worked out.

In this paper we give a new method to obtain the finite action based on the Gibbs-Duhem relation and Gibbs free energy for NUT solutions. According to this method, all thermodynamic quantities in arbitrary even dimensions can be obtained. The validity of the first law of thermodynamics of black holes has been tested for Taub-NUT spacetimes in Einstein gravity [9, 12]. We also check the validity of this law in EGB gravity.

The outline of our paper is as follows. We give a brief review of the general formalism of EGB gravity, the counterterm method and Taub-NUT/Bolt spacetime in Section II. In Section III, two possible Taub-NUT/Bolt solutions of EGB gravity in six dimensions are reviewed. Then, in sections IV and V, all thermodynamic quantities for NUT and Bolt solutions are obtained and thermodynamical stability is investigated. finaly, in Section VI, a generalization of the thermodynamics to $2k + 2$-dimensional Taub-NUT-AdS spacetimes in EGB gravity is made. We finish the paper with some concluding remarks.

II. GENERAL FORMALISM

The gravitational action of EGB gravity in d dimension is

$$I_G = \frac{1}{16\pi} \int_{\mathcal{M}} d^dx \sqrt{-g}[-2\Lambda + R + \alpha L_{GB}] \tag{1}$$

where Λ is the cosmological constant; R, $R_{\mu\nu\rho\sigma}$, and $R_{\mu\nu}$ are the Ricci scalar, the Riemann and the Ricci tensors of the metric $g_{\mu\nu}$; and α is the GB coefficient with dimension of (length)^2. L_{GB} is the GB term,

$$L_{GB} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{2}$$

In order to have a well-defined variational principle, we must consider two surface terms, the Gibbons-Hawking term and its counterpart for GB gravity, which are

$$I_b^{(E)} = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} K, \tag{3}$$

$$I_b^{(GB)} = -\frac{\alpha}{4\pi} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma}(J - 2G_{ij}K^{ij}), \tag{4}$$

where γ_{ij} is the induced metric on the boundary, K is the trace of the extrinsic curvature of the boundary, G_{ij} is the Einstein tensor on the induced metric, and J is the trace of the tensor,

$$J_{ab} = \frac{1}{3}(2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}). \tag{5}$$

We restrict ourselves to the case of $\alpha > 0$, which is consistent with heterotic string theory [28]. Since this term is a topological invariant in four dimensions, we therefore apply this term to higher dimensions which consist of only second order derivatives of the metric that produces second order field equations. Varying the action with respect to the metric tensor $g_{\mu\nu}$, the vacuum field equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H_{\mu\nu} = 0, \tag{6}$$

where

$$H_{\mu\nu} = 2RR_{\mu\nu} - 4R_{\mu\lambda}R^\lambda_{\ \nu} - 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} + 2R_\mu^{\ \rho\sigma\lambda}R_{\nu\rho\sigma\lambda} - \frac{1}{2}g_{\mu\nu}L_{GB}. \tag{7}$$

We want to consider asymptotic AdS spacetime which has negative scalar curvature at infinity. This implies the following asymptotic expression for the Riemann tensor:

$$R_{\mu\nu}^{\ \ \lambda\sigma} = -\frac{\delta_\mu^\lambda\delta_\nu^\sigma - \delta_\mu^\sigma\delta_\nu^\lambda}{\ell_c^2}, \tag{8}$$

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where ℓ_c is the effective radius of AdS spacetime in EGB gravity [22]. This parameter is

$$\ell_c = \sqrt{\frac{2\alpha(d-3)(d-4)}{1-U}}, \tag{9}$$

where

$$U = \sqrt{1 + \frac{8\Lambda\alpha(d-3)(d-4)}{(d-1)(d-2)}} \tag{10}$$

with $\Lambda = -(d-1)(d-2)/2\ell^2$. It is seen from Eq. (10) that the parameter α must have an upper bound as

$$\alpha \leq \alpha_{\max} = \frac{(d-1)(d-2)}{8\,|\Lambda|\,(d-3)(d-4)}. \tag{11}$$

In the computation of the total action $I_G + I_b^E + I_b^{GB}$ and conserved charges of EGB solutions, one encounters infrared divergences, associated with the infinite volume of the spacetime manifold. This quantities are computed by using a counterterm method which was proposed by Balasubramanian and Kraus [20] for AdS-Einstein gravity. The counterterm that regularizes the action for $d < 8$ solutions, for EGB gravity [22] is

$$\begin{aligned}
I_{\text{ct}}^{\text{EGB}} &= \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} \{ -(\frac{d-2}{\ell_c})(\frac{2+U}{3}) - \frac{\ell_c \Theta(d-4)}{2(d-3)}(2-U)R \\
&\quad - \frac{\ell_c^3 \Theta(d-6)}{2(d-3)^2(d-5)}[U(R_{ab}R^{ab} - \frac{d-1}{4(d-2)}R^2) - \frac{d-3}{2(d-4)}(U-1)L_{GB}^{(in)}]\},
\end{aligned} \tag{12}$$

where R, R^{ab} and $L_{GB}^{(in)}$ are the curvature, the Ricci tensor and the GB term (2) associated with the induced metric γ. Also, $\Theta(x)$ is the step-function with $\Theta(x) = 1$ provided that $x \geq 0$, and zero otherwise. It can be seen that as $\alpha \to 0$ ($U \to 1$), this action reduces to the familiar counterterm expression in the Einstein gravity [20].

Having the total finite action, one can construct a divergence-free stress-energy tensor by varying the total action with respect to the boundary metric γ_{ab}, (Brown and York's definition of energy-momentum tensor [29]). The total finite stress-energy tensor for $d < 8$ [22] is

$$\begin{aligned}
8\pi T_{ab} &= \frac{16\pi}{\sqrt{-\gamma}} \frac{\delta}{\delta\gamma^{ab}} \left(I_G + I_b^{\text{E}} + I_b^{\text{GB}} + I_{\text{ct}}^{\text{EGB}}\right) = K_{ab} - \gamma_{ab}K + 2\alpha(Q_{ab} - \frac{1}{3}Q\gamma_{ab}) \\
&\quad - \frac{d-2}{\ell_c}\gamma_{ab}(\frac{2+U}{3}) + \frac{\ell_c \Theta(d-4)}{d-3}(2-U)\left(R_{ab} - \frac{1}{2}\gamma_{ab}R\right) + \ell_c^3 \Theta(d-6) \\
&\quad \{\frac{U}{(d-3)^2(d-5)}[-\frac{1}{2}\gamma_{ab}(R_{cd}R^{cd} - \frac{(d-1)}{4(d-2)}R^2) - \frac{(d-1)}{2(d-2)}RR_{ab} + 2R^{cd}R_{cadb} \\
&\quad - \frac{d-3}{2(d-2)}\nabla_a\nabla_b R + \nabla^2 R_{ab} - \frac{1}{2(d-2)}\gamma_{ab}\nabla^2 R] - \frac{U-1}{2(d-3)(d-4)(d-5)}H_{ab}\} + \ldots,
\end{aligned} \tag{13}$$

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where [30]

$$Q_{ab} = 2KK_{ac}K_b^c - 2K_{ac}K^{cd}K_{db} + K_{ab}(K_{cd}K^{cd} - K^2) + 2KR_{ab} + RK_{ab} - 2K^{cd}R_{cadb} - 4R_{ac}K_b^c, \tag{14}$$

and H_{ab} is given by (7) in terms of the boundary metric γ_{ab}.

From the finite stress-energy tensor, one can compute conserved charges. To do this job, we must choose a spacelike surface Σ in the boundary $\partial \mathcal{M}$ with metric σ_{ij} and write the boundary metric in the Arnowitt-Deser-Misner form,

$$\gamma_{ab}dx^a dx^b = -\mathcal{N}^2 dt^2 + \sigma_{ij}(d\varphi^i + \mathcal{V}^i dt)(d\varphi^j + \mathcal{V}^j dt), \tag{15}$$

where $\mathcal{N}$ and $\mathcal{V}^i$ are the lapse function and shift vector, respectively. The coordinates φ^i, $i = 1, \ldots, d-2$ are the angular variables, parameterizing the hypersurface of constant r. The conserved charge associated with a killing vector ξ^a is

$$\mathfrak{Q}(\xi) = \oint_{\Sigma} d^{d-2}x \sqrt{\sigma} u^a T_{ab} \xi^b, \tag{16}$$

where σ is the determinant of the metric σ_{ij} and u^a is the normal to quasilocal boundary hypersurface Σ. For example the conserved mass M, is the charge associated with the timelike killing vector $\xi = \partial_t$.

The Taub-NUT/Bolt-AdS metric is one of the solutions of EGB gravity which was investigated in $(2k+2)$ dimensions [4]. This metric is constructed on a base space endowed with an Einstein-Kähler metric $\Xi_{\mathcal{B}}$. The Euclidean section of the $(2k+2)$-dimensional Taub-NUT/Bolt spacetime can be written as:

$$ds^2 = F(r)(d\tau + NA)^2 + F^{-1}(r)dr^2 + (r^2 - N^2)\Xi_{\mathcal{B}} \tag{17}$$

where τ is the coordinate on the fibre S^1 and A is the Kähler form of the base space B, which is proportional to some covariantly constant two-form. Here N is the NUT charge and $F(r)$ is a function of r which is obtained by solving the EGB field equation (6). The solution will describe a 'NUT' if the fixed point set of the U(1) isometry $\partial/\partial\tau$ (i.e. the points where $F(r) = 0$) is less than $2k$-dimension and a 'Bolt' if the fixed point set is $2k$-dimensions.

III.   SIX-DIMENSIONAL SOLUTIONS

In this section we give a revision of the six-dimensional Taub-NUT/bolt solutions(17) of GB gravity will be done. The function $F(r)$ for all nonextremal choices of the base space $\mathcal{B}$

can be written [4] in the form of

$$F(r) = \frac{(r^2 - N^2)^2}{12\alpha(r^2 + N^2)} \left(1 + \frac{4\alpha}{(r^2 - N^2)} - \sqrt{B(r) + P(r)} \right) \tag{18}$$

where

$$B(r) = 1 + \frac{16\alpha N^2(r^4 + 6r^2N^2 + N^4) + 12\alpha mr(r^2 + N^2)}{(r^2 - N^2)^4}$$
$$+ \frac{12\alpha\Lambda(r^2 + N^2)}{5(r^2 - N^2)^4}(r^6 - 5N^2r^4 + 15N^4r^2 + 5N^6) \tag{19}$$

and the function $P(r)$ depends on the choice of the curved base space $\mathcal{B}$. For the case $\mathcal{B} = \mathbb{CP}^2$, the Kähler form A and the $\mathbb{CP}^2$ metric are

$$A_2 = 6\sin^2\xi_2(d\psi_2 + \sin^2\xi_1 d\psi_1) \tag{20}$$

$$d\Sigma_2{}^2 = 6\{d\xi_2{}^2 + \sin^2\xi_2\cos^2\xi_2(d\psi_2 + \sin^2\xi_1 d\psi_1)^2$$
$$+ sin^2\xi_2(d\xi_1{}^2 + \sin^2\xi_1\cos^2\xi_1 d\psi_1{}^2)\}. \tag{21}$$

The function $P(r)$ in this case is

$$P_{\mathbb{CP}^2} = -\frac{16\alpha^2(r^4 + 6r^2N^2 + N^4)}{(r^2 - N^2)^4}. \tag{22}$$

The other possibility is $\mathcal{B} = S^2 \times S^2$, where S^2 is the 2-sphere with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and the one-form A is

$$A = 2\cos\theta_1 d\phi_1 + 2\cos\theta_2 d\phi_2. \tag{23}$$

In this case $P(r)$ is

$$P_{S^2 \times S^2} = -\frac{32\alpha^2(r^4 + 4r^2N^2 + N^4)}{(r^2 - N^2)^4}. \tag{24}$$

For asymptotically AdS spacetimes [$\Lambda < 0$ provided that $|\Lambda| < 5/(12\alpha_{\max})$], the function $F(r)$ is not real for all values of r in the range $0 \le r \le \infty$ and $\alpha > 0$.

The Ricci scalar and L_{GB} in the bulk (1) are

$$R = -\frac{1}{(r^2 - N^2)^2}\frac{d}{dr}\left[\frac{dF(r)}{dr}(r^2 - N^2)^2 + 4rF(r)(r^2 - N^2) - \frac{4}{3}(r^2 - 3N^2)\right], \tag{25}$$

$$L_{GB} = \frac{8}{(r^2 - N^2)^2}\frac{d}{dr}\left\{\frac{dF(r)}{dr}\left[3F(r)(r^2 + N^2) - (r^2 - N^2)\right] + 3rF(r)^2 - 2rF(r) + \Gamma r\right\}, \tag{26}$$

where $\Gamma = 2/3$ for $\mathcal{B} = \mathbb{CP}^2$ and "1" for $\mathcal{B} = S^2 \times S^2$ respectively.

A. NUT Solutions: In brief, the conditions for having a NUT solution, which have been described completely in some previous works [9, 12], are as follows:

(i) $F(r = N) = 0$; (ii) $F'(r = N) = 1/(3N)$; (iii) $F(r)$ should have no positive roots at $r > N$. The first condition comes from the fact that all extra dimensions should collapse to zero at the fixed point set of $\partial/\partial t$. The second condition allows one to avoid a conical singularity with a smoothly closed fiber at $r = N$. Using the first condition, the GB gravity may admit NUT solutions with $\mathbb{CP}^2$ and $S^2 \times S^2$ base spaces when the mass parameter m is fixed to be

$$m_n^{(\mathbb{CP})} = -\frac{16}{15} N(3\Lambda N^4 + 5N^2 - 5\alpha), \tag{27}$$

$$m_n^{(\mathbb{S})} = -\frac{8}{15} N(6\Lambda N^4 + 10N^2 - 15\alpha). \tag{28}$$

By inserting m_n from Eqs. (27) and (28) in (18), the function $F_n(r)$ can be obtained. But as it has been mentioned in Ref. [4], the metric with $\mathcal{B} = S^2 \times S^2$ has a curvature singularity at $r = N$. Although the second condition to have a NUT solution is satisfied numerically, by having a curvature singularity at $r = N$, this condition is basically violated. Therefore, the metric does not admit a NUT solution in this case. In Fig. 1, the function $F_n(r)$ is plotted versus r for $\alpha = 1/35$ and $\Lambda = -10$ for both base spaces. As it is shown in this figure, at the small vicinity of $r = N$, $F_n(r)$ is complex with a continues real part. Following section IV, we will show that if one does not consider this fact and investigate the thermodynamics of NUT solutions for spacetimes with both base spaces, no obvious difference will be seen. In the limit of $\alpha \to 0$, $F_n(r)$ for both base factors goes to F_{NUT} in Einstein gravity [9]. The allowed range of $F(r)$ in the $\mathbb{CP}^2$ case is more extended than $S^2 \times S^2$ case. Also if α goes to α_{max}, or $u \to 0$, $F_n(r)$ would be complex in some range of r so that in both base spaces, we find that $F_n(r)$ is not real at $\alpha = \alpha_{max}$ for all values of r in the range of $0 \le r < \infty$ and $\alpha > 0$. This fact is true for NUT solutions of arbitrary even dimensions.

The boundary of this kind of spacetime will be an asymptotic surface at some large radius r. The boundary metric γ_{ij} diverges at infinity. By rescaling γ_{ab} upon a conformal factor ℓ_c^2/r^2 (see Eq.(9)), the dual boundary metric $h_{ab} = \lim\limits_{r \to \infty} \ell_c^2/r^2 \gamma_{ab}$ converges and the general line element of the dual field theory for even-dimensional TNAdS spacetimes on the boundary can be obtained as

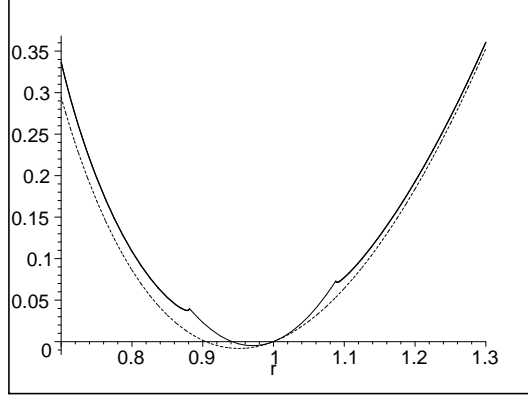$$ds_b^2 = (d\tau + NA)^2 + \ell_c^2 \Xi_{\mathcal{B}}. \tag{29}$$

FIG. 1: $F_n(r)$ versus r for $\mathcal{B} = \mathbb{CP}^2$ (dotted line) and $\mathcal{B} = S^2 \times S^2$ ("bold-line" for pure real and "solid-line" for the real part in the vicinity of $N = 1$).

In fact, $\lim_{r\to\infty} \ell_c^2/r^2 F_{NUT}(r) = 1$. As it has been shown in (29), the Gauss-Bonnet coefficient α enters the metric on the boundary due to ℓ_c.

B. Bolt Solutions: The conditions for having Bolt Solutions are (i) $F(r = r_b) = 0$ and (ii) $F'(r_b) = 1/(3N)$ with $r_b > N$. These Conditions give the following relation for m_b

$$m_b^{(\mathbb{CP})} = -\frac{3\Lambda(r_b^6 + 5N^6) - 10(r_b^4 - 3N^4) + 15N^2 r_b^2[4 - \Lambda(r_b^2 - 3N^2)] - 40\alpha(r_b^2 + N^2)}{15 r_b} \quad (30)$$

and an equation for r_b with the base space $\mathbb{CP}^2$

$$3N\Lambda r_b{}^3 + (2 + 3\Lambda N^2) r_b{}^2 - N(4 + 3\Lambda N^2) r_b - 3\Lambda N^4 - 6N^2 + 8\alpha = 0 \quad (31)$$

which at least has one real solution. This real solution for $N < \sqrt{\alpha}$ yields $r_b > N$.

For the case of $\mathcal{B} = S^2 \times S^2$, the mass parameter is

$$m_b^{(S)} = -\frac{3\Lambda(r_b^6 + 5N^6) - 10(r_b^4 - 3N^4) + 15N^2 r_b^2[4 - \Lambda(r_b^2 - 3N^2)] - 60\alpha(r_b^2 + N^2)}{15 r_b}, \quad (32)$$

and r_b can be found by solving the following equation:

$$3N\Lambda r_b{}^4 - 6N(\Lambda N^2 + 1) r_b{}^2 + 2(r_b^2 - N^2) r_b + 3(\Lambda N^2 + 2)N^3 + 4\alpha(2r_b - 3N) = 0. \quad (33)$$

This equation will have two real roots if $N \leq N_{\max}$. At $N = N_{\max}$ there will be only one $r_b > N$. Therefore in order to have Bolt solution in this case, there is a maximum value for N [4]. It is found that the allowed range of r_b for this case is more extended than in the $\mathbb{CP}^2$ case.

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IV. THERMODYNAMICS OF 6-D NUT SOLUTIONS

A. $\mathbb{CP}^2$ case: From relations (25) and (26), we can simply compute the bulk action (1). By computing the other actions for the boundary and counterterm (3), (4), and (12), the finite action $I_{\text{fin}} = I_G + I_b^{(E)} + I_b^{(GB)} - I_{\text{ct}}^{(EGB)}$ can be obtained as

$$I_{\text{fin(N)}}^{(\mathbb{CP}^2)} = -\frac{8\pi N\beta}{15}(2\Lambda N^4 + 5N^2 - 10\alpha). \tag{34}$$

In Eq. (44), $\beta = 4\pi/F'(N) = 12\pi N$ is the inverse of the Hawking temperature T_H which is found by demanding regularity of the Euclidean manifold (17). The total mass can be calculated as

$$\mathcal{M}_N = 2\pi m_n. \tag{35}$$

The total angular momentum is zero, which is the same as Einstein gravity. The Gibbs free energy and entropy may be computed as

$$G_N^{(\mathbb{CP}^2)}(T_H) = \frac{I}{\beta} = -\frac{\Lambda + 360\pi^2 T_H^2}{233280\pi^4 T_H^5} + \frac{4\alpha}{9T_H}, \tag{36}$$

$$S_N^{(\mathbb{CP}^2)} = -(\frac{\partial G}{\partial T_H}) = -32\pi^2 N^2(2\Lambda N^4 + 3N^2 - 2\alpha), \tag{37}$$

Also the entropy can be obtained, as the Einstein case [9], by using Gibbs-Duhem relation, which is

$$S = \beta(\mathcal{M} - \mu_i \mathcal{C}_i) - I \tag{38}$$

where μ_i is chemical potential and $\mathcal{C}_i$ is the conserved charge (such as mass, angular momentum,...). The Smarr type formula for mass versus extensive quantity S may be written as

$$\mathcal{M}_N^{(\mathbb{CP}^2)}(S) = -\frac{32\pi\sqrt{Z}}{15}(3\Lambda Z^2 + 5Z - 5\alpha) \tag{39}$$

where $Z = N^2$ is a function of S, which may be obtained from the following equation:

$$S + 32\pi^2 Z(2\Lambda Z^2 + 3Z - 2\alpha) = 0. \tag{40}$$

The Hawking temperature T_H may be calculated as

$$T_H = \frac{d\mathcal{M}_N(S)}{ds} = \frac{1}{12\pi N} \tag{41}$$

Now the specific heat can be obtained as

$$C_N^{(\mathbb{CP}^2)} = T_H(\frac{\partial S}{\partial T_H}) = 128\pi^2 N^2(3\Lambda N^4 + 3N^2 - \alpha). \tag{42}$$

Note that the entropy become negative for $\alpha < 3/2N^2 - |\Lambda| N^4$ and specific heat is negative for $\alpha > 3(N^2 - |\Lambda| N^4)$. Thus, for thermally stable solution for $\mathcal{B} = \mathbb{CP}^2$ case, the value of α parameter must be in the range of

$$3/2N^2 - |\Lambda| N^4 \leq \alpha \leq 3(N^2 - |\Lambda| N^4), \tag{43}$$

provided that $\alpha \leq \alpha_{\max}$, $N > 0$ and $\Lambda < 0$. In the limit of $\alpha \to 0$, (Einstein gravity), this spacetime is completely unstable [12].

B. $S^2 \times S^2$ case: As in the previous case, the finite action, Gibbs free energy, entropy, and specific heat may be obtained as

$$I_{\text{fin(N)}}^{(S^2 \times S^2)} = -\frac{8\pi N\beta}{15}(2\Lambda N^4 + 5N^2 - 15\alpha), \tag{44}$$

$$G_N^{(S^2 \times S^2)}(T_H) = \frac{I}{\beta} = -\frac{\Lambda + 360\pi^2 T_H^2}{233280\pi^4 T_H^5} + \frac{2\alpha}{3T_H} \tag{45}$$

$$S_N^{(S^2 \times S^2)} = -32\pi^2 N^2(2\Lambda N^4 + 3N^2 - 3\alpha). \tag{46}$$

$$C_N^{(S^2 \times S^2)} = 192\pi^2 N^2(2\Lambda N^4 + 2N^2 - \alpha). \tag{47}$$

By comparing all quantities in the $\mathcal{B} = \mathbb{CP}^2$ and the $S^2 \times S^2$ case, we can see that by substituting $\alpha^{\mathbb{CP}^2} = (3/2)\alpha^{S^2 \times S^2}$ in all equations in the previous case, all thermodynamic quantities can be obtained in this base space. By expanding computations in $2k + 2$ dimensions, it can be seen that

$$\alpha^{\mathbb{CP}^k} = \frac{k+1}{k}\alpha^{S^2 \times S^2 \times \dots S_k^2}. \tag{48}$$

According to this fact, there is no difference between these two base spaces and if we do not consider the metric with $S^2 \times S^2$, as a NUT solution of EGB gravity, nothing would be missed.

It is worthwhile to mention that, for all thermodynamic quantities which are calculated in these cases, the first law of thermodynamics, $d\mathcal{M} = T_H dS$, is satisfied.

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V. THERMODYNAMICS OF 6-D BOLT SOLUTIONS

A. $S^2 \times S^2$ case: The finite action can be calculated as

$$I_b^{(S^2 \times S^2)} = \frac{4\pi^2}{15[2(r_b^2 - N^2) - \Lambda(r_b^2 - N^2)^2 + 4\alpha]} \{(15\Lambda N^8 + 30 N^6$$
$$+ 30 N^4 r_b^2 - 70 N^2 r_b^4 - 10 N^4 \Lambda r_b^4 - 8 N^2 \Lambda r_b^6 + 10 r_b^6$$
$$+ 3 r_b^8 \Lambda) + (120 N^4 r_b^2 \Lambda - 300 N^4 + 100 r_b^4 - 120 N^2 r_b^2$$
$$- 200 N^2 r_b^4 \Lambda - 120 \Lambda N^6 + 72 \Lambda r_b^6)\alpha + 480(N^2 + r_b^2)\alpha^2\}. \tag{49}$$

In the limit of $\alpha \to 0$ the action (49) goes to the corresponding action in Einstein gravity [9, 12]. The entropy and specific heat may be written as

$$S_b^{(S^2 \times S^2)} = \frac{4\pi^2}{3[2(r_b^2 - N^2) - \Lambda(r_b^2 - N^2)^2 + 4\alpha]} \{\Lambda(16 n^2 r_b^6 - 3 r_b^8$$
$$+ 24 n^6 r_b^2 + 9 n^8 - 46 n^4 r_b^4) + 6(3 n^6 + r_b^6) - 42 n^2 r_b^4 + 18 n^4 r_b^2$$
$$+ 96 (n^2 + r_b^2)\alpha^2 + (-84 n^4 - 24 \Lambda n^6 - 168 n^2 r_b^2 - 24 \Lambda r_b^6$$
$$+ 60 r_b^4 + 88 n^2 \Lambda r_b^4 - 168 n^4 \Lambda r_b^2)\alpha\}, \tag{50}$$

$$C_b^{(S^2 \times S^2)} = \frac{8\pi^2(r_b^2 - N^2 + 4\alpha)}{3[\alpha(8\Lambda r_b^2 - 8\Lambda N^2 - 4) + \Lambda(r_b^2 - N^2)^2][\Lambda(r_b^2 - N^2)^2 - 2(r_b^2 - N^2) - 4\alpha]}$$
$$\{ - 192\alpha^3 + (576 r_b^2 \Lambda N^2 - 32 \Lambda r_b^4 + 336 N^2 + 96 \Lambda N^4 + 144 r_b^2) \alpha^2$$
$$+ (20 r_b^8 \Lambda^2 - 12 \Lambda^2 N^8 + 376 N^2 r_b^4 \Lambda + 48 N^6 r_b^2 \Lambda^2 - 120 \Lambda N^6$$
$$- 144 N^2 r_b^6 \Lambda^2 + 192 r_b^4 + 96 r_b^2 N^2 - 72 N^4 r_b^2 \Lambda - 120 r_b^6 \Lambda - 192 N^4$$
$$+ 88 N^4 r_b^4 \Lambda^2)\alpha - 88 N^4 \Lambda r_b^4 + 104 N^2 \Lambda r_b^6 + 36 N^6 + 36 r_b^6 + 36 N^8 \Lambda$$
$$+ 74 r_b^6 \Lambda^2 N^4 - 30 \Lambda^2 N^6 r_b^4 - 15 \Lambda^2 N^8 r_b^2 - 43 r_b^8 \Lambda^2 N^2 - 36 r_b^2 N^4$$
$$- 36 r_b^4 N^2 - 28 r_b^8 \Lambda + 5 r_b^{10} \Lambda^2 + 9 \Lambda^2 N^{10} - 24 r_b^2 \Lambda N^6\} \tag{51}$$

In Fig. 2 the entropy and specific heat are plotted for $\Lambda = -10$ and $\alpha = 1/30$ as a function of N in the allowed range of $N < N_{\max} = 0.12$, for the outer horizon of the Bolt solution. As it can be see from this figure, the Bolt solution in this case is completely stable.

B. $\mathbb{CP}^2$ case: The finite action, entropy, and heat capacity can be obtained as

$$I_b^{(\mathbb{CP}^2)} = \frac{8\pi^2(r_b^2 - N^2 + 4\alpha)}{5[3\Lambda(r_b^2 - N^2)^2 - 6(r_b^2 - N^2) - 8\alpha]} \{ (-40 r_b^2 - 40 N^2) \alpha$$
$$- 40 N^3 r_b - 32 N^5 r_b \Lambda + 15 \Lambda N^6 + 45 N^4 r_b^2 \Lambda + 30 N^4$$
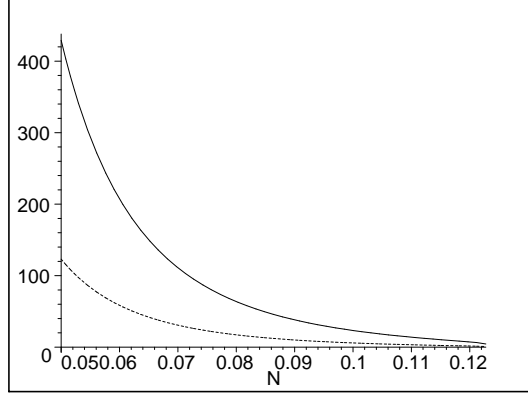$$- 10 r_b^4 - 15 N^2 r_b^4 \Lambda + 60 r_b^2 N^2 + 3 r_b^6 \Lambda\} \tag{52}$$

FIG. 2: Entropy (dot) and specific heat (bold-line) versus N of Bolt solution for $\mathcal{B} = S^2 \times S^2$.

$$
\begin{aligned}
S_b^{(\mathbb{CP}^2)} &= \frac{8\pi^2(r_b^2 - N^2 + 4\alpha)}{5[3\Lambda(r_b^2 - N^2)^2 - 6(r_b^2 - N^2) - 8\alpha]}\{ \left(-40\,r_b{}^2 - 40\,N^2\right)\alpha \\
&\quad + 15\,\Lambda\,N^6 + 45\,N^4 r_b{}^2\Lambda + 30\,N^4 - 10\,r_b{}^4 - 15\,N^2 r_b{}^4\Lambda \\
&\quad + 60\,r_b{}^2 N^2 + 32\,N^5 r_b\Lambda + 3\,r_b{}^6\Lambda + 40\,N^3 r_b\}
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
C_b^{(\mathbb{CP}^2)} &= \frac{16\pi^2(r_b^2 - N^2 + 4\alpha)}{5[8\alpha(3\Lambda r_b^2 - 3\Lambda N^2 - 2) + 3\Lambda(r_b^2 - N^2)^2][3\Lambda(r_b^2 - N^2)^2 - 6(r_b^2 - N^2) - 8\alpha]} \\
&\quad \{ -180\,r_b{}^2\Lambda\,N^6 + 450\,N^3 r_b{}^5\Lambda - 630\,N^5 r_b{}^3\Lambda + 180\,r_b{}^5 N - 300\,N^4\Lambda\,r_b{}^4 \\
&\quad + 420\,N^2\Lambda\,r_b{}^6 + 270\,N^7 r_b\Lambda - 90\,\Lambda^2 N^8 r_b{}^2 - 120\,r_b{}^8\Lambda - 360\,N^3 r_b{}^3 + \\
&\quad + 180\,N^6 + 180\,r_b{}^6 + 180\,N^8\Lambda + 45\,\Lambda^2 N^{10} + 312\,N^5 r_b{}^5\Lambda^2 + 180\,N^5 r_b \\
&\quad - 264\,N^7 r_b{}^3\Lambda^2 - 90\,\Lambda^2 N^6 r_b{}^4 - 171\,r_b{}^8\Lambda^2 N^2 + 18\,r_b{}^{10}\Lambda^2 + 72\,N^9 r_b\Lambda^2 \\
&\quad + 288\,r_b{}^6\Lambda^2 N^4 - 180\,r_b{}^2 N^4 - 180\,r_b{}^4 N^2 - 90\,r_b{}^7 N\Lambda - 120\,r_b{}^7 N^3\Lambda^2 \\
&\quad + 10\left(128\,N^2 + 48\Lambda\,r_b{}^4 + 96 N r_b + 128 N^3 r_b\Lambda + 96 r_b{}^2\Lambda\,N^2 + 48\Lambda\,N^4\right)\alpha^2 \\
&\quad + (-90\,\Lambda^2 N^8 + 480\,N^4 r_b{}^2\Lambda - 640\,N^3 r_b - 840\,N^4 + 54\,r_b{}^8\Lambda^2 - 480\,N^3 r_b{}^5\Lambda^2 \\
&\quad - 360\,N r_b{}^5\Lambda + 1520\,r_b{}^3 N^3\Lambda + 960\,r_b{}^3 N + 720\,r_b{}^2 N^2 + 360\,N^6 r_b{}^2\Lambda^2 \\
&\quad - 600\,\Lambda\,N^6 - 504\,N^2 r_b{}^6\Lambda^2 + 180\,N^4 r_b{}^4\Lambda^2 + 840\,N^2 r_b{}^4\Lambda - 96\,N^7 r_b\Lambda^2 \\
&\quad - 336\,r_b{}^6\Lambda + 576\,N^5 r_b{}^3\Lambda^2 + 760\,r_b{}^4 - 904\,N^5 r_b\Lambda)\alpha - 640\,\alpha^3\}.
\end{aligned}
\tag{54}
$$

Fig. 3 shows the entropy and specific heat for $\Lambda = -10$ and $\alpha = 1/30$ as a function of N in the allowed range. It is shown that the Bolt solution in this case is completely unstable.

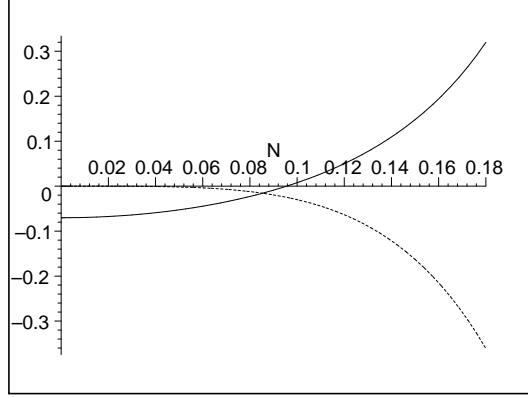## VI.   THERMODYNAMICS OF $(2k+2)$-DIMENSIONAL NUT SOLUTIONS

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FIG. 3: Entropy (dot) and specific heat (bold-line) versus N of Bolt solution for $\mathcal{B} = \mathbb{CP}^2$.

In this section a new procedure to achieve thermodynamic quantities by using the Gibbs-Duhem relation and Gibbs free energy will be introduced.

The generalized form of even-dimensional TNAdS spacetimes in EGB gravity has been constructed [4]. The mass parameter in $d = 2k+2$ dimensions for $\mathcal{B} = \mathbb{CP}^k$ can be calculated as

$$
\begin{aligned}
m_N^k &= \frac{(-1)^{k+1} k 4^{k+1} N^{2k-3}}{\Gamma(2k+3)} (\Gamma(k))^2 \\
&\quad \left[2\Lambda N^4 (k+1)^2 + (2k^2 + 3k + 1)kN^2 + (1 - 4k^2)(k-1)k\alpha \right].
\end{aligned} \tag{55}
$$

As it has been mentioned in $6D$ NUT solutions, all quantities go to the Einstein gravity case in the limit of $\alpha \to 0$. Also the GB parameter α, has not been appear in the calculation of the total mass and inverse Hawking temperature β. The similar case has been shown in Ref [31]. Therefore we may consider the total mass and β, similar to d-dimensional NUT solution in Einstein gravity [12] which have been obtained as

$$
\beta^k = 4(k+1)\pi N, \tag{56}
$$

$$
\mathcal{M}^k = \frac{k(4\pi)^{k-1} m}{4}. \tag{57}
$$

What about finite action? Up to now, the main method for computing the finite action has been the counterterm method, which has been used in Sec. IV for six dimensions. In order to extend this method to higher dimensions, one needs more terms than (12). However recently, a new counterterm action has been achieved for $7 < d \leq 9$ in EGB gravity which has so many terms [22]. According to this method, the existence of the counterterm action is necessary for computing the thermodynamic quantities. Now we give a different method

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based on the Gibbs-Duhem relation (38) and Gibbs free energy for NUT solutions that can calculate regularized action independent of the counterterm method.

According to this method, first, first write the finite action in $2k+2$-dimensional Einstein gravity [12] with an additional term as a function of α. This term must be considered in such a way that in the limit of $\alpha \to 0$, it goes to zero for consistency to Einstein gravity. Therefore the finite action in EGB gravity can be written as

$$
\begin{aligned}
I_N^k &= \frac{4^k \pi^{(k-3/2)} N^{2\,k-3} \beta}{32(k+1)} \Gamma(k+1) \Gamma(-1/2-k) \\
&\quad \left[(2\,k^2 + 3\,k+1)N^2 + (k+1)2\,\Lambda\,N^4 + \Theta(k-2)\,f_k(\alpha, N) \right].
\end{aligned}
\tag{58}
$$

After that, calculate the Gibbs free energy as a function of T_H and then calculate the entropy S_G similar to Eq. (37). The other way of computing the entropy, which is independent of the previous method, is to use the Gibbs-Duhem relation (38). By calculating the entropy in the latter way (S_{GD}), the first order differential equation $S_{GD} = S_G$ gives us $f_k(\alpha, N)$ as

$$
f_k(\alpha, N) = -k(4k^2 - 1)\alpha.
\tag{59}
$$

The results in $6D$ may be considered as an initial condition. The function (59) is independent of N and is the same as the last term in (55). Now the finite action may be calculated in EGB gravity by substituting (59) in the action (58). For $k = 2$ ($6D$), the action (58) is equal to (34) in section IV. In the appendix, computations for $8D$ are done using the counterterm method and we see that there is a good coincidence with our prescription. The Gibbs free energy, entropy, and heat capacity may be obtained as

$$
\begin{aligned}
G_N^k(T_H) &= \frac{\Gamma(k+1)\,\Gamma(-1/2-k)}{((k+1)\,T_H)^{2\,k+1}\,\pi^{k+5/2}4^{k+3}} [8\,\pi^2 T_H{}^2 \left(5\,k^2 + 2\,k^3 + 4\,k+1\right) \\
&\quad -128\left(4\,k^5 + 12\,k^4 + 11\,k^3 + k^2 - 3\,k - 1\right)\pi^4 T_H{}^4 k\alpha + \Lambda],
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
S_N^k &= -4^{k-1}\pi^{(k-1/2)} N^{2\,k-2}\Gamma(1/2-k)\,\Gamma(k+1) \\
&\quad \left[(2\,k^2 + k - 1)N^2 + (k+1)2\,\Lambda\,N^4 - (4k^2 - 8k+3)k\alpha \right],
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
C_N^k &= 4^{(k-1/2)}\pi^{k-1/2} N^{2\,k-2}\Gamma(1/2-k)\,\Gamma(k+1) \\
&\quad \left[(2\,k^2 + k - 1)kN^2 + (1+2k)2\,\Lambda\,N^4 - (4\,k^3 - 12\,k^2 + 11\,k - 3)k\alpha \right].
\end{aligned}
\tag{62}
$$

The spacetime for $\mathcal{B} = \mathbb{CP}^k$ is stable in the following range:

$$
\frac{N^2[2(k+1)N^2\Lambda + (2k^2 + k - 1)]}{k(4k^2 - 8k + 3)} \leq \alpha \leq \frac{N^2[2(k+1)^2 N^2\Lambda + k(2k^2 + k - 1)]}{k(4k^3 - 12k^2 + 11k - 3)}.
\tag{63}
$$

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provided that $0 < \alpha \le \alpha_{\max}$ (11), $N > 0$, and $\Lambda < 0$. According to all calculations in this section, the validity of the first law of thermodynamics of black holes can be simply verified.

VII. CONCLUSION

In this paper, a review of the existence of non extremal asymptotically AdS Taub-NUT/Bolt solutions in EGB gravity with curved base spaces has been discussed. The spacetime with base space $S^2 \times S^2$ has a curvature singularity at $r = N$. This fact violates the second condition for having a NUT solution. Therefore, the metric basically does not admit NUT solutions. But, as this condition has been satisfied numerically, we may proceed with computations the same as in the $\mathbb{CP}^2$ case. We show that in the limit of $\alpha \to 0$, $F_n(r)$ for both base spaces, goes to F_{NUT} in Einstein gravity. Also if α goes to α_{max}, or $u \to 0$, $F_n(r)$ would be complex in some range of r so that in both base spaces, we find that $F_{n(r)}$ is not real at $\alpha = \alpha_{max}$ for all values of r in the range $0 \le r < \infty$ and $\alpha > 0$. This fact is true for NUT solutions of arbitrary even dimensions. The boundary of Taub-NUT-AdS spacetime at some large radius r is obtained, and it was shown that the Gauss-Bonnet coefficient α enters the metric on the boundary due to ℓ_c. The investigation of the thermodynamics of NUT/bolt solutions in six dimensions is carried out. By obtaining the regularized action and stress tensor, the total mass is derived. The Smarr-type formula for the mass as a function of the extensive parameter S is calculated. By computing the Hawking temperature, the validity of the first law of thermodynamics is demonstrated. It is shown that in NUT solutions all thermodynamic quantities for both base spaces are related to each other by substituting $\alpha^{\mathbb{CP}^k} = [(k+1)/k]\alpha^{S^2 \times S^2 \times \cdots S_k^2}$. This relation is not true for bolt solutions. So, no further information is given by investigating NUT solutions in the $S^2 \times S^2$ case, and we can remove this metric from NUT solutions. A generalization of the thermodynamics of black holes in arbitrary even dimensions is made using a new method based on the Gibbs- Duhem relation and Gibbs free energy for NUT solutions. According to this method, the finite action for TNAdS spacetimes in EGB gravity is obtained by considering the generalized finite action for TNAdS spacetimes in Einstein gravity with an additional term as a function of α. This term should vanished if $\alpha \to 0$. The great importance of this method is the computation of the finite action, which is easier than the counterterm prescription, for some spacetimes higher than seven dimensions in higher derivative gravity. However, this is just a conjecture

that its validity must be checked as an open problem.

Finally, we perform the stability analysis by investigating the heat capacity and entropy in the allowed range of α, Λ, and N. For NUT solutions in d dimensions, there exists a stable phase at a narrow range of α for both base factors. For bolt solutions in six dimensions, the metric is completely stable in the allowed range of α and N for the base factor $\mathcal{B} = S^2 \times S^2$, and is completely unstable for $\mathcal{B} = \mathbb{CP}^2$ case. An interesting open problem would be to compare the results of different prescriptions (the counterterm method, Kounterterm, and our procedure) in computing the finite action.

APPENDIX: FINITE ACTION FOR EIGHT-DIMENSIONAL TNADS SPACETIMES IN EGB GRAVITY

The Ricci scalar and L_{GB} for $d = 8$ in the bulk are:

$$R = -\frac{1}{(r^2 - N^2)^3}\frac{d}{dr}\left[\frac{dF(r)}{dr}\left(r^2 - N^2\right)^3 + 6\,rF(r)\left(r^2 - N^2\right)^2 - 6\,rN^4 + 4\,r^3N^2 - \frac{6}{5}r^5\right],$$
(64)

$$\begin{aligned}L_{GB} &= \frac{12}{(r^2 - N^2)^3}\frac{d}{dr}\left[\frac{dF(r)}{dr}F(r)\left(5\,r^4 - 2\,r^2N^2 - 3\,N^4\right) - \frac{dF(r)}{dr}\left(r^2 - N^2\right)^2\right.\\ &\left.\quad + 2\,rF(r)^2\left(5\,r^2 - N^2\right) - 4\,rF(r)\left(r^2 - N^2\right) + \frac{1}{2}r\left(r^2 - 3N^2\right)\right].\end{aligned}$$
(65)

In This case counterterm has been obtained [22] as

$$\begin{aligned}I_{ct} &= \frac{1}{8\pi}\int_{\partial\mathcal{M}}d^{d-1}x\sqrt{-\gamma}\left\{-(\frac{d-2}{\ell_c})(\frac{2+U}{3}) - \frac{\ell_c\Theta(d-4)}{2(d-3)}(2-U)R\right.\\ &\quad -\frac{\ell_c^3\Theta(d-6)}{2(d-3)^2(d-5)}\left[U(R_{ab}R^{ab} - \frac{d-1}{4(d-2)}R^2) - \frac{d-3}{2(d-4)}(U-1)L_{GB}^{(in)}\right]\\ &\quad +\Theta(d-8)\left[\left(1 + 31/30(U-1)\right)L_E - \frac{19\ell_c^5}{57600}(U-1)\mathcal{L}_{(3)}^{(in)}\right]\right\},\end{aligned}$$
(66)

where

$$\begin{aligned}L_E &= \frac{\ell_c^5}{(d-3)^3(d-5)(d-7)}\left(\frac{3d-1}{4(d-2)}RR^{ab}R_{ab} - \frac{(d-1)(d+1)}{16(d-2)^2}R^3\right.\\ &\quad \left. - 2R^{ab}R^{cd}R_{acbd} + \frac{d-3}{2(d-2)}R^{ab}\nabla_a\nabla_b R - R^{ab}\nabla^2 R_{ab} + \frac{1}{2(d-2)}R\nabla^2 R\right),\end{aligned}$$
(67)

and the third Lovelock term is

$$\begin{aligned}\mathcal{L}_{(3)}^{(in)} &= 2R^{abcd}R_{cdef}R^{ef}{}_{ab} + 8R^{ab}{}_{cd}R^{ce}{}_{bf}R^{df}{}_{ae} + 24R^{abcd}R_{cdbe}R^e{}_a\\ &\quad + 3RR^{abcd}R_{cdab} + 24R^{abcd}R_{ca}R_{db} + 16R^{ab}R_{bc}R^c_a - 12RR^{ab}R_{ab} + R^3.\end{aligned}$$
(68)

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All quantities in (66), (67) and (68) must be calculated in the induced metric γ_{ab}. By considering all actions in the Bulk, boundary, and counterterm, the finite action can be obtained as

$$I_{\text{fin(N)}}^{(\mathbb{CP}^3)} = \frac{16\pi^2 N^3 \beta}{35}(8\Lambda N^4 + 28N^2 - 105\alpha).\tag{69}$$

It has a good coincidence with the action (58) for $k = 3$ in Section (VI).

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